

# Numerical Studies of Hamiltonian Systems and Application to Galactic Potentials

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## 1 Talk Summary

The talk consisted mainly in commenting in a linear way the Hénon & Heiles (1964) paper. Instead of repeating here the lecture of the paper, we advise the reader interested in dynamical systems to study this “must” reading. Below are a few comments added during the talk and references.

Michel Hénon’s contact with Geneva Observatory started in the 70’s when the third course of the Saas Fee series was organized by Louis Martinet and Michel Mayor including Michel Hénon as speaker (Hénon 1973), together with Donald Lynden-Bell and Georges Contopoulos. Around this time Louis Martinet, my thesis supervisor, inspired by Michel Hénon worked at understanding the chaos in galactic potentials (e.g., Martinet 1974), which led naturally to my thesis topic, the dynamics of barred galaxies, which are typical systems where chaos and regular motion coexist each in substantial parts.

All this activity was for good part consequence of the seminal paper in 1964 by Michel Hénon and graduate student Carl Heiles at Princeton University: “*The applicability of the third integral of motion: Some numerical experiments*” in the field of galactic dynamics. Although the incentive is galactic dynamics, the scope of this paper is much broader. Together with the paper of Edward Lorentz “Deterministic Nonperiodic Flow” (1963) these two papers were pivotal in launching a new field of research called “dynamical system theory”; Hénon & Heiles paper for the domain of Hamiltonian dynamical systems, and Lorentz’ paper for the domain of dissipative systems. Both papers were motivated by modeling concrete scientific questions with computers (the third integral in galactic potentials and the long-term weather prediction, respectively). Before these papers, dynamical systems were widely believed to be either completely integrable or completely ergodic. After, the possibility of *semi-ergodic* motion (a very appropriate adjective used by Hénon but later replaced in popularity by “chaotic”) provided a continuum of possibilities between the two integrable or ergodic extreme cases, which represent actually a measure zero subset in the set of all dynamical systems. Chaos is characterized by sensitive dependence on

initial conditions and also often occurs close to resonances. As argued in Hénon & Heiles paper, if in a given dynamical system a range of initial conditions show a transition from regular quasi-periodic to chaotic motion occurs, often the transition is sharp.

The Hénon & Heiles paper was promoting the use of computers to perform numerical experiments as a proper research method. An earlier example in this area of this kind was the famous Fermi, Pasta & Ulam paper (1955) trying precisely to understand the transition from quasi-periodic to ergodic in a 1-dimensional chain of non-linear oscillators. The Hénon & Heiles paper was also promoting the use of the clever tool of the surface of section in dynamical systems, invented by Poincaré long before (1899) but not much used in numerical works. Hénon brought a step further the application of this tool, replacing the time-continuous Hamiltonian system by a time-discrete iterated Hamiltonian mapping, gaining by a factor  $\sim 1000$  in computing power if the objective is just to study *typical*, in this case, Hamiltonian systems. I used precisely this trick in my thesis for understanding the phase space neighborhood of complex unstable periodic orbits, first by using 4D Hamiltonian mappings and then applying the knowledge to galactic orbits (Pfenniger 1985ab).

At the same time, the mathematical understanding of Hamiltonian systems increased much through the KAM theorem (Kolmogorov 1954; Arnold 1963; Moser 1962). By using numerical experiments Michel Hénon brought much clarity to the KAM theorem meaning for concrete dynamical problems to the physical and astronomical communities. Without such an insight brought by computer plots, probably the KAM results would have stood confined to the mathematical community for many decades, very much like the multi-decade old idea of fractal sets which was popularized some years later by Benot Mandelbrot (e.g., Mandelbrot 1977) using also simple computer models and graphical representations.

The Hénon & Heiles paper, currently with about 1000 citations, is the most cited paper of Michel Hénon's list, due to its breath and innovative content touching fundamental aspects of classical mechanics. It reached its citation rate peak in 1985, and since then the citation rate slowly decreases, which characterizes a pioneer and seminal paper, two decades in advance over the scientific community main preoccupations, and still of lasting value 50 years later.

My only curiosity frustration bears on the background work methods unexplained in the paper, probably for the sake of clarity. For example, several numerical or procedural clever methods had to be invented with the early 60's computers, in particular to elaborate the plots. For instance, the elegant method for cleanly computing surface of sections was only explained much later by Hénon (Hénon 1982), although computer codes including it were circulating well before.

In conclusion, the Hénon & Heiles paper is exemplary in many ways and introduce several innovations. It is clear, concise and pedagogical. It is exemplary especially on the method used by Hénon throughout his research: to simplify a problem as much as possible to gain in generality, yet keeping the non-trivial

properties that make the problem difficult.

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